

ON KÄHLER-EINSTEIN SURFACES WITH EDGE SINGULARITIES

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ABSTRACT. In this paper we characterize logarithmic surfaces which admit Kähler-Einstein metrics with negative scalar curvature and small edge singularities along a normal crossing divisor.

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1. INTRODUCTION

In a celebrated paper [27], S.-T. Yau solved the Calabi conjecture by studying complex Monge-Ampère equations on compact Kähler manifolds.

The solution of the Calabi conjecture provides a general existence theorem for *Kähler-Einstein* metrics of negative or zero scalar curvature (in the negative case, independently, T. Aubin [2]).

Since then, complex Monge-Ampère equations have been extensively studied. In particular, major developments in the theory of complex Monge-Ampère equations with singular right hand side have occurred, see for example [4], [20], [11]. As in the non-degenerate case, these analytical results provide very general existence theorems for *singular* Kähler-Einstein metrics. For more details, results and the connection with the theory of the minimal model see also [13] and the bibliography therein.

Such existence theorems, although extremely general, provide little control on the asymptotic behavior of the singular Kähler-Einstein metric near the *degeneracy* locus. This fact somehow limits many of the geometric applications.

In [25], G. Tian proposed to look for Kähler-Einstein metrics with *cone* singularities along a normal crossing divisor. Such metrics should still arise as solutions of certain singular complex Monge-Ampère equations, but because of the mild type of singularities developed by the associated Kähler-Einstein potentials one should be able to derive many interesting geometric consequences, see again [25].

In light of the recent advances and interest in the theory of Kähler-Einstein metrics with edge singularities [12], [7], [18], [1], we study the geometry of logarithmic surfaces which admit Kähler-Einstein metrics of negative scalar curvature and edge singularities of small cone angles along a normal crossing divisor.

The paper is organized as follows. In Section 2, we use the theory of Kähler currents to derive some generalities regarding the existence of Kähler-Einstein metrics with edge singularities. In Section 3, we show how the theory of *semi-stable* curves on algebraic surfaces developed by F. Sakai in [24] is relevant for the study of Kähler-Einstein metrics with edges on algebraic surfaces. Section 4 contains a proof of the geometric semi-positivity of certain twisted log-canonical bundles associated to a logarithmic surface. Finally, in Section 5, we show how the results in [7] and [18] can be used to classify logarithmic surfaces which admit negative Kähler-Einstein metrics with *small* edge singularities. Moreover, we briefly discuss the Chern-Weil approach, through Kähler-Einstein metrics with cone-edge singularities, to the logarithmic *Bogomolov-Miyaoka-Yau* inequality for surfaces of log-general type. Remarkably, this is intimately connected with the recent work of M. F. Atiyah and C. LeBrun in [1].

2. KÄHLER-EINSTEIN METRICS WITH EDGE SINGULARITIES AS KÄHLER CURRENTS

Let \overline{M} be a n -dimensional projective manifold and D a normal crossing divisor. Given the pair (\overline{M}, D) , let us define the notion of a Kähler metric with edge singularities along D .

Let $\{D_i\}$ be the irreducible components of D and $\{\alpha_i\}$ a collection of positive numbers less than one and bigger than zero. A smooth Kähler metric $\hat{\omega}$ on $\overline{M} \setminus D$ is said to have edge singularities of cone angles $2\pi(1 - \alpha_i)$ along the D_i 's if for any point $p \in D$ there exist a coordinate neighborhood $(\Omega; z_1, \dots, z_n)$ where

$$D|_{\Omega} = z_1 \cdot \dots \cdot z_k = 0$$

and a positive constant C such that

$$(1) \quad C^{-1}\omega_0 \leq \hat{\omega} \leq C\omega_0$$

where

$$(2) \quad \omega_0 = \sqrt{-1} \left(\sum_{i=1}^k \frac{dz_i \wedge d\bar{z}_i}{|z_i|^{2\alpha_i}} + \sum_{i=k+1}^n dz_i \wedge d\bar{z}_i \right).$$

Summarizing, from the Kähler geometry point of view on $\overline{M} \setminus D$ the *smooth* form $\hat{\omega}$ is simply an *incomplete* Kähler metric with finite volume. Because of the finite volume property, standard results in the theory of currents [10] imply that $\hat{\omega}$ can be regarded as a strictly positive closed real $(1, 1)$ -current on \overline{M} . In other words, $\hat{\omega}$ is a *Kähler current* on \overline{M} with *singular support* D .

We are now ready to introduce the definition of a Kähler-Einstein metric with edge singularities.

Definition 1 (Tian [25]). *A Kähler current $\hat{\omega}$ with edges of cone angles $2\pi(1 - \alpha_i)$ along the D_i 's is called Kähler-Einstein with curvature λ if it satisfies the distributional equation*

$$(3) \quad Ric_{\hat{\omega}} - \sum_i \alpha_i [D_i] = \lambda \hat{\omega}$$

where by $[D]$ we indicate the current of integration along D .

In what follows, we will focus on the negatively curved case. Thus, let $\hat{\omega}$ be a Kähler-Einstein metrics with edge singularities as in 3 with $\lambda = -1$. By the Poincaré-Lelong formula [15], the current $-Ric_{\hat{\omega}} + \sum_i \alpha_i [D_i]$ represents the cohomology class of the \mathbb{R} -divisor $K_{\overline{M}} + \sum_i \alpha_i D_i$. Since by assumption $\hat{\omega}$ satisfies 3 with $\lambda = -1$, we have that the cohomology class of the \mathbb{R} -divisor $K_{\overline{M}} + \sum_i \alpha_i D_i$ can be represented by a Kähler current. By the structure of the *pseudo-effective* cone given in [9], we conclude that $K_{\overline{M}} + \sum_i \alpha_i D_i$ is a *big* \mathbb{R} -divisor.

Next, we want to show that $K_{\overline{M}} + \sum_i \alpha_i D_i$ has to be an *ample* \mathbb{R} -divisor. To this aim, recall that given a closed positive $(1, 1)$ -current T in \mathbb{C}^n one can define the Lelong numbers [15]. More precisely, let

$$\nu(T, r, x) = \frac{1}{(\pi r^2)^{n-1}} \int_{B(x, r)} T \wedge \omega^{n-1}$$

where ω is the standard Euclidean Kähler form. The Lelong number of T at x is then simply defined as

$$\nu(T, x) := \lim_{r \rightarrow 0} \nu(T, r, x).$$

The theory of positive currents, as developed by Lelong, Siu and others see for example [10], ensures that the definition given above makes sense and that furthermore it can be extended to currents on Kähler manifolds.

We are now interested in the computation of Lelong numbers of a Kähler current with edge singularities. For simplicity we treat the complex surface case only. The general case is completely analogous.

Let D^* denote the smooth part of the divisor D . Because of the *quasi-isometric* condition given in 1, for any $x \in D^*$ the computation of the Lelong numbers of $\hat{\omega}$ reduces to the evaluation of

$$\lim_{r \rightarrow 0} \int_{B(r)} \frac{1}{|z_1|^{2\alpha}} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2.$$

Since

$$\int \frac{dr}{r^{2\alpha-1}} = \frac{1}{2(1-\alpha)} r^{2(1-\alpha)}$$

and $0 < \alpha < 1$, we conclude that $\nu(\hat{\omega}, x) = 0$ for any $x \in D^*$. Furthermore, the Lelong numbers are clearly zero outside the singular support of $\hat{\omega}$.

We are now ready to show that $\hat{\omega}$ represents an ample class. The idea is to apply the celebrated *regularization* result of Demailly [9].

For cohomological reasons, on a compact Kähler manifold, the Lelong numbers of a given positive current are bounded from above. Let us then choose a positive number c such that $\nu(\hat{\omega}, x) < c$ for any point $x \in \overline{M}$. Next, observe that there exists a smooth positive $(1, 1)$ -form γ such that $\hat{\omega} \geq \gamma$. Because of the choice of the number c , the regularizations $\{\hat{\omega}_{c,k}\}$ given in the main theorem of [9] are smooth over \overline{M} and satisfy

$$\hat{\omega}_{c,k} \geq \gamma - \mu_{c,k} \bar{\omega}$$

where $\{\mu_{c,k}\}$ is decreasing sequence of continuous functions converging pointwise to $\nu(\hat{\omega}, x)$, and $\bar{\omega}$ is a given background Kähler metric. As a result, given any

irreducible curve C on \overline{M} , we have

$$(\hat{\omega}, C) = (\hat{\omega}_{c,k}, C) = \int_{C^*} \hat{\omega}_{c,k} \geq \int_{C^*} \gamma - \int_{C^*} \mu_{c,k} \bar{\omega}$$

where by C^* we denote the smooth part of C . Letting k go to infinity and applying the dominated convergence theorem of Lebesgue, we conclude that $\hat{\omega}$ represents a *strictly* nef class. By a Nakai-Moishezon type criterion for \mathbb{R} -divisors due to F. Campana and T. Peternell [8], we conclude that $K_{\overline{M}} + \sum_i \alpha_i D_i$ is indeed an ample \mathbb{R} -divisor.

Let us summarize these observations into a proposition.

Proposition 2.1. *Let $\hat{\omega}$ be a Kähler-Einstein metric with edge singularities as in \mathcal{B} with $\lambda = -1$, then the \mathbb{R} -divisor $K_{\overline{M}} + \sum_i \alpha_i D_i$ is ample.*

An analogous proposition holds for Kähler-Einstein currents of positive scalar curvature.

Proposition 2.2. *Let $\hat{\omega}$ be a Kähler-Einstein metric with edge singularities as in \mathcal{B} with $\lambda = 1$, then the \mathbb{R} -divisor $-(K_{\overline{M}} + \sum_i \alpha_i D_i)$ is ample.*

For the zero scalar curvature case the cohomological restrictions are obvious.

3. ON KÄHLER-EINSTEIN SURFACES WITH SMALL EDGE SINGULARITIES AND NEGATIVE SCALAR CURVATURE

Let \overline{M} be a smooth projective surface. Let D be a reduced divisor having normal crossings on \overline{M} . The *logarithmic* canonical bundle associated to D is then defined as $\mathcal{L} = K_{\overline{M}} + D$, for details see [17]. Similarly, for any real number $\alpha \in (0, 1]$ we define $\mathcal{L}_\alpha = K_{\overline{M}} + \alpha D$. Let us refer to the \mathcal{L}_α 's as *twisted log-canonical bundles*.

In Section 2, we have shown that if a pair (\overline{M}, D) admits a Kähler-Einstein metric with edge singularities of cone angle $2\pi(1 - \alpha)$ along D then the \mathbb{R} -divisor \mathcal{L}_α has to be ample, see Proposition 2.1. In this section we classify the pairs (\overline{M}, D) for which the corresponding \mathcal{L}_α 's are ample for all α close enough to one. Furthermore, we give an analogous classification result under the weaker requirement for \mathcal{L}_α to be big and nef for all α close enough to one. This more general classification result will be used in Section 4 and in the applications given in Section 5. Let us start with the following definitions, for details see [24].

Definition 2. *Let \overline{M} be an algebraic surface and let D be a reduced normal crossing divisor. The pair (\overline{M}, D) is said to be D -minimal if there are no curves E in \overline{M} such that*

$$E \simeq \mathbb{CP}^1, \quad E^2 = -1, \quad E \cdot D \leq 1.$$

Observe that if (\overline{M}, D) is D -minimal then \overline{M} need not be minimal.

Definition 3. *The divisor D is called semi-stable if any smooth rational component in D intersects the other components of D at least in two points.*

We can now prove the following lemma.

Lemma 3.1. *If \mathcal{L}_α is big and nef for all values of α close enough to one, then the pair (\overline{M}, D) is D -minimal of log-general type with D a semi-stable curve.*

Proof. Let us assume \mathcal{L}_α to be big and nef for all values of α close to one. Let consider the limit $\mathcal{L}_\alpha \rightarrow \mathcal{L}$ as α approaches one. By Kleiman's theorem [21], the closure of the *ample* cone is exactly the *nef* cone. In particular, the nef cone is closed and \mathcal{L} is necessarily nef. Let E be an irreducible component of D , we then have

$$\mathcal{L} \cdot E = 2p_a(E) - 2 + E(D - E)$$

and since by construction \mathcal{L} is nef, we conclude that D is *semi-stable*. Moreover, the pair (\overline{M}, D) is D -minimal. Finally, it remains to show that \mathcal{L} is a big divisor. Since

$$\mathcal{L} = \mathcal{L}_\alpha + (1 - \alpha)D$$

and by assumption \mathcal{L}_α is big, the result simply follows from the structure of the algebraic pseudo-effective cone [21]. \square

Conversely, we can prove the following.

Lemma 3.2. *Let (\overline{M}, D) be a D -minimal semi-stable pair of log-general type, then \mathcal{L}_α is big and nef for all values of α close to one.*

Proof. Since by assumption \mathcal{L} is big and the set of big \mathbb{R} -divisors forms an open convex cone, \mathcal{L}_α is big for all values of α close enough to one, see Corollary 2.24. in [21].

It remains to understand the *nefness* properties of \mathcal{L}_α . Let E be an irreducible divisor in \overline{M} and let us write

$$(4) \quad \mathcal{L}_\alpha \cdot E = \mathcal{L} \cdot E + (\alpha - 1)D \cdot E.$$

Now, the semi-stability assumption on D combined with the D -minimality of (\overline{M}, D) imply that \mathcal{L} is nef. Thus, if E is such that $\mathcal{L} \cdot E > 0$, by making α close enough to one we can always assume that $\mathcal{L}_\alpha \cdot E > 0$. The only curves we then need to consider are the ones such that $\mathcal{L} \cdot E = 0$. Thus, if E is irreducible and such that $\mathcal{L} \cdot E = 0$ we have the following cases:

- $E \not\subset D$, $E \simeq \mathbb{CP}^1$, $E^2 = -2$;
- $E \subset D$, E is an isolated component of D with $p_a(E) = 1$, or $E \simeq \mathbb{CP}^1$ and $E \cdot (D - E) = 2$.

Let us study first the case when E is an isolated component of D . Observe that

$$\mathcal{L} \cdot E = 2p_a(E) - 2 + D \cdot E - E^2$$

so that if $\mathcal{L} \cdot E = 0$ and $p_a(E) = 1$ then $D \cdot E = E^2$. Since $\mathcal{L}^2 > 0$, by the Hodge index theorem we conclude that $D \cdot E = E^2 < 0$. By using 4, we conclude that \mathcal{L}_α dots positively with the isolated components of D with arithmetic genus equal to one.

Let now E be a (-2) -curve not contained in the boundary divisor D such that $\mathcal{L} \cdot E = 0$. Since in this case $D \cdot E = 0$ we have

$$\mathcal{L}_\alpha \cdot E = \mathcal{L} \cdot E + (\alpha - 1)D \cdot E = 0$$

constantly in α .

Finally, let E be a smooth rational component of D such that $E \cdot (D - E) = 2$ and $\mathcal{L} \cdot E = 0$. Again by the Hodge index theorem we must have $E^2 < 0$. As a result

$$\mathcal{L}_\alpha \cdot E = (\alpha - 1)D \cdot E = (\alpha - 1)(2 + E^2) \geq 0$$

with equality iff E is a (-2) -curve. \square

Combining Lemma 3.1 and 3.2 we have then proved the following theorem.

Theorem 3.3. *The \mathbb{R} -divisor \mathcal{L}_α is big and nef for all values of α close enough to one iff the pair (\overline{M}, D) is D -minimal of log-general type with D a semi-stable curve.*

Finally, we characterize when \mathcal{L}_α is ample for all values of α close to one.

Theorem 3.4. *The \mathbb{R} -divisor \mathcal{L}_α is ample for all values of α close to one iff the pair (\overline{M}, D) is D -minimal of log-general type without interior rational (-2) -curves and D is a semi-stable curve without (-2) -rational curves intersecting the other components of D in just two points.*

Proof. Let us assume \mathcal{L}_α to be ample for all values of α close to one. By Lemma 3.1 the pair (\overline{M}, D) is minimal of log-general type and D semi-stable. Finally, it is clear that the pair (\overline{M}, D) cannot contain rational (-2) -curves as above.

Conversely, if (\overline{M}, D) is as in statement, the ampleness of \mathcal{L}_α follows from the computations in Lemma 3.2 and the characterization of the amplitude for \mathbb{R} -divisors given in [8]. \square

4. GEOMETRIC SEMI-POSITIVITY OF TWISTED LOG-CANONICAL BUNDLES

In Section 3 we have shown that given a minimal semi-stable log-general pair (\overline{M}, D) there are obstructions for the \mathbb{R} -divisors \mathcal{L}_α to be ample for all α close enough to one. Nevertheless one expects good positivity properties for these \mathbb{R} -divisors.

Recall the following definition, see for example [14].

Definition 4. *A line bundle L is called geometrically semi-positive if $c_1(L)$ can be represented by a smooth closed Hermitian $(1, 1)$ -form which is everywhere positive semi-definite.*

In this section we want to show that, given a minimal semi-stable log-general pair (\overline{M}, D) , the associated \mathbb{R} -divisor \mathcal{L}_α is geometrically semi-positive for all values of α close enough to one. Moreover, we will precisely describe the locus where these \mathbb{R} -divisors fail to be ample.

The main tool used here will be a well-known theorem of Reider [23]. In fact, we apply this theorem to certain integer multiples of \mathcal{L}_α where the parameter α is appropriately chosen to be rational and close enough to one.

Thus, let us start with \mathbb{Q} -divisors of the form

$$\mathcal{L}_{\alpha_n} = K_{\overline{M}} + \frac{(n-2)}{n}D.$$

By clearing denominators we obtain

$$(5) \quad n\mathcal{L}_{\alpha_n} = 2K_{\overline{M}} + (n-2)\mathcal{L} = K_{\overline{M}} + (K_{\overline{M}} + (n-2)\mathcal{L}).$$

By construction the pair (\overline{M}, D) is D -minimal of log-general type and D a semi-stable pair. The log-canonical bundle \mathcal{L} is then big and nef. Furthermore, for n big enough by Theorem 3.3 the divisor $\overline{\mathcal{L}}_n = K_{\overline{M}} + (n-2)\mathcal{L}$ is big and nef.

Now, given a big and nef divisor L on \overline{M} the theorem of Reider [23] provides a powerful tool for the study of the linear system $|K_{\overline{M}} + L|$. The idea is now to apply this theorem to the linear system associated to the divisor given in 5.

By letting n be big enough, we can always assume that $\overline{\mathcal{L}}_n^2 > 4$. By the theorem of Reider, we know that if $x \in \overline{M}$ is a base point of $|K_{\overline{M}} + \overline{\mathcal{L}}_n|$ then there exists an effective divisor C such that $x \in C$ and

$$(6) \quad \overline{\mathcal{L}}_n \cdot C = 0, \quad C^2 = -1; \quad \overline{\mathcal{L}}_n \cdot C = 1, \quad C^2 = 0.$$

For n big enough, the divisor C in 6 must satisfy $\mathcal{L} \cdot C = 0$. By the Hodge index theorem we then have $C^2 < 0$. This rule out the second possibility in 6. Regarding the remaining case, we argue as follows. Since $\mathcal{L} \cdot C = 0$, the divisor C must be connected. Furthermore, since $C^2 = -1$ it is easy to see that such a divisor must be reduced. But then we would have

$$K_{\overline{M}} \cdot C = 0, \quad C^2 = -1,$$

which contradicts the integrality of the *arithmetic genus* of C [16]. Concluding, for n big enough the linear system $|n\mathcal{L}_{\alpha_n}|$ is base-point free. In other words, the Kodaira map

$$i_{|K_{\overline{M}} + \overline{\mathcal{L}}_n|} : \overline{M} \longrightarrow P^N$$

is everywhere defined for n big enough. We then have that \mathcal{L}_{α_n} can be represented by a smooth closed Hermitian $(1, 1)$ -form which is everywhere positive semi-definite. Next, we want to understand the *locus* where \mathcal{L}_{α_n} fails to be a Kähler class. Again this can be achieved by using Reider's theorem. More precisely, for n big enough the only obstruction for the Kodaira map $i_{|K_{\overline{M}} + \overline{\mathcal{L}}_n|}$ to be a local diffeomorphism onto its image is given by the existence of effective divisors C such that

$$\mathcal{L} \cdot C = 0, \quad K_{\overline{M}} \cdot C = 0, \quad C^2 = \{-1, -2\}.$$

These divisors are now easily classified. In fact, by using the reasoning given in the proof of Lemma 3.2, we conclude that the only obstructions are given by the interior (-2) -curves and the (-2) -curves in D intersecting the other components of D in two points only.

Proposition 4.1. *Let (\overline{M}, D) be D -minimal of log-general type with D a semi-stable curve. There exists $\overline{\alpha} \in (0, 1)$ such that for any $\alpha \in [\overline{\alpha}, 1)$ then \mathcal{L}_α can be represented by a smooth closed $(1, 1)$ -form which is everywhere positive semi-definite and strictly positive outside the interior (-2) -curves and the boundary (-2) -curves intersecting the other components of D in just two points.*

Proof. We have seen that, for n big enough, there exists a representative for \mathcal{L}_{α_n} which is everywhere positive semi-definite and strictly positive outside the interior (-2) -curves and the boundary (-2) -curves intersecting the other components of D in just two points. By Theorem 5.8. in [24] we know that \mathcal{L} is *semi-ample*. We then have that \mathcal{L} can be represented by a positive semi-definite smooth form. A simple computation now shows that, for any $\alpha \in [\alpha_n, 1)$, there exist strictly positive real numbers $\beta_1(\alpha)$ and $\beta_2(\alpha)$ such that

$$\mathcal{L}_\alpha = \beta_1 \mathcal{L}_{\alpha_n} + \beta_2 \mathcal{L}.$$

By letting $\bar{\alpha}$ be equal to α_n , the proof is then complete. \square

5. APPLICATIONS

In this section, we apply the recent analytical advances in the theory of complex Monge-Ampère equations with degenerate right hand side [8], [18] to classify logarithmic surfaces which admit Kähler-Einstein metrics with negative scalar curvature and small edge singularities.

As in Section 2, let D be a normal crossing divisor and let $\{D_i\}$ be its irreducible components. For all i , let denote by L_i the line bundle associated to D_i and let $\sigma_i \in H^0(\bar{M}, \mathcal{O}_{\bar{M}}(L_i))$ be a defining section for D_i . Finally, equip each of these line bundles with a Hermitian metric $\{(L_i, \|\cdot\|)\}$.

Thus, if we are interested in constructing singular negative Kähler-Einstein metrics on $\bar{M} \setminus D$ with asymptotic behavior as in 2, given a Kähler class ω on \bar{M} , it is natural to consider the following singular complex Monge-Ampère equation

$$(7) \quad (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{f+\varphi} \frac{\omega^n}{\prod_i \|\sigma_i\|^{2\alpha}}$$

whose right hand side is the volume form of an edge metric with cone angle $2\pi(1-\alpha)$.

In fact, if we assume \mathcal{L}_α to be ample by choosing $\omega \in [\mathcal{L}_\alpha]$ and f such that

$$(8) \quad \sqrt{-1}(\partial\bar{\partial} \log \omega^n - \sum_i \alpha_i \partial\bar{\partial} \log \|\sigma_i\|^2 + \partial\bar{\partial} f) = \omega$$

if φ is a solution of 7, smooth outside D , it is clear that ω_φ is a smooth Kähler-Einstein metric with negative scalar curvature on $\bar{M} \setminus D$.

Equations of the type given in 7 were already studied in the fundamental paper of S.-T. Yau [27]. The approach described in [27] is through the study of non-singular ϵ -regularization of 7. More precisely, one tries to construct a solution of 7 by studying the degeneration as $\epsilon \rightarrow 0$ of the solutions of ϵ -regularized equations of the form

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi_\epsilon)^n = e^{f+\varphi_\epsilon} \frac{\omega^n}{\prod_i (\|\sigma_i\|^2 + \epsilon^2)^\alpha},$$

for more details see Section 8 in [27].

In [20], [19], S. Kołodziej using techniques coming from *pluripotential* theory, developed a very general existence, uniqueness and regularity theory for complex Monge-Ampère equations whose right hand side is a L^p -density for some $p > 1$. In particular, this very general theory can be applied to solve equations like 7 for $\alpha \in (0, 1)$.

Finally, over the past year there has been a lot of progress towards the completion of Tian's program [25]. In fact R. Mazzeo, T. Jeffres and Y. Rubinstein in [18], building up on the work of S. Donaldson [12], appear to have completed this program in all dimensions, for all cone angles when D is an irreducible *smooth* divisor. Mazzeo-Rubinstein announced the resolution of this problem in the general case when D has simple normal crossings [22]. These results make use of Rubinstein's Ricci continuity method and Mazzeo's edge calculus which in particular provides a fine asymptotic for the associated Kähler-Einstein potentials, for more details see [18] and the bibliography therein. Moreover, they have existence theorems when $\mathcal{L}_\alpha \sim 0$ and $-\mathcal{L}_\alpha$ is ample. For related results see also the works of R. Berman [5] and S. Brendle [6]. Interestingly, by using an approach similar to the one suggested

by S.-T. Yau in [27], F. Campana, H. Guenancia, M. Păun in [7] were able to show the existence of a negatively curved Kähler-Einstein metric with edges along a normal crossing divisor D under the assumption that \mathcal{L}_α is ample and $\alpha \in [\frac{1}{2}, 1)$. We now use their main existence result, Theorem A in [7] or alternatively Theorem 1.3. in [22], to prove the following.

Theorem 5.1. *A logarithmic surface (\overline{M}, D) admits negative Kähler-Einstein metrics with edge singularities along D and cone angles $2\pi(1 - \alpha)$ for all values of α close enough to one iff (\overline{M}, D) is D -minimal, log-general, D is a semi-stable curve and there are no interior (-2) -curves or (-2) -curves in D which intersects the other components of D in two points only.*

Proof. By Proposition 2.1, we know that if (\overline{M}, D) admits a negative Kähler-Einstein metric with edge singularities along D with cone angle $2\pi(1 - \alpha)$ then the associated twisted log-canonical bundle \mathcal{L}_α has to be ample. Thus, if \mathcal{L}_α is ample for all values of α close to one by Theorem 3.4 we conclude that (\overline{M}, D) is minimal, log-general, D is a semi-stable curve and there are no interior (-2) -curves or (-2) -curves in D which intersects the other components of D in just two points.

Conversely, if (\overline{M}, D) is as above by Theorem 3.4 we know that the \mathcal{L}_α 's are ample for all values of α close to one. Let us choose a Kähler class ω in $[\mathcal{L}_\alpha]$ and f as in 8, then by solving a singular complex Monge-Ampère equation of the form given in 7, see for example [20], [7], we can construct a negative Kähler-Einstein metric ω_φ on $\overline{M} \setminus D$. Now, since we are working with values of α close to one we can assume $\alpha \in [\frac{1}{2}, 1)$ and then applying Theorem A in [7] or Theorem 1.3. in [22] we conclude that ω_φ is indeed *quasi-isometric* to an edge Kähler metric near D . \square

We conclude this section by discussing the Bogomolov-Miyaoka-Yau inequality for surfaces of log-general type. In [26], G. Tian and S.-T. Yau were able to prove, among many other things, that given a logarithmic surface (\overline{M}, D) for which \mathcal{L} is big, nef and ample modulo D then the inequality

$$c_1^2(\Omega_{\overline{M}}^1(\log D)) \leq 3c_2(\Omega_{\overline{M}}^1(\log D))$$

holds. For the definition of the sheaf $\Omega_{\overline{M}}^1(\log D)$ and its basic properties we refer to Chapter 3 in [15]. As the reader can easily verify, if (\overline{M}, D) is log-general, D -minimal with D a semi-stable curve and there are not interior (-2) -curves then \mathcal{L} is big, nef and ample modulo D . Following [25], one may try to prove 9 by deforming to zero the cone angle of a family ω_φ^α of negative Kähler-Einstein metrics with edges singularities along D and apply a suitably modified Chern-Weil theory for this kind of incomplete metrics. The theory developed by Jeffress-Mazzeo-Rubinstein, see Theorem 2 in [18], provides a precise asymptotic for ω_φ near D . Remarkably, this asymptotic behavior appears to be exactly what is needed in order to develop a meaningful Chern-Weil theory in this context. In fact, Atiyah and LeBrun in [1] introduce the notion of Riemannian *edge-cone* metrics which are singular along smoothly embedded codimension two submanifolds and derive the analogues of the well-known *Gauss-Bonnet* and *signature* formulas for closed 4-manifolds. As the reader can easily verify, if the cone angle is sufficiently small, the singular Kähler-Einstein metrics constructed by Jeffress-Mazzeo-Rubinstein have cone-edge singularities in the sense of Atiyah-LeBrun. Again this follows from the deep analysis contained in Theorem 2. of [18]. Thus, let us show that this approach does indeed work when the boundary divisor D is *smooth*.

Proposition 5.2. *Let (\overline{M}, D) be D -minimal of log-general type without interior (-2) -curves and let D be a smooth semi-stable curve. Then*

$$(9) \quad c_1^2(\Omega_{\overline{M}}^1(\log D)) \leq 3c_2(\Omega_{\overline{M}}^1(\log D)).$$

Proof. By Theorem 3.4, if (\overline{M}, D) is as in the statement then the twisted log-canonical bundle \mathcal{L}_α is ample for all values of α close enough to one. By solving a singular Monge-Ampère equation as in 7, see again Theorem 2. in [18], one can construct a family of singular Kähler-Einstein metrics ω_φ^α with negative scalar curvature and cone angles $2\pi(1 - \alpha)$ along D . For α close enough to one, by Theorem 2.1. and Theorem 2.2. in [1] we know that

$$\chi(M, \omega_\varphi^\alpha) = \chi(\overline{M}) - \alpha\chi(D), \quad \sigma(M, \omega_\varphi^\alpha) = \sigma(\overline{M}) - \frac{1}{3}\alpha(2 - \alpha)D^2$$

where $M = \overline{M} \setminus D$. For simplicity let us define

$$\chi_\alpha = \chi(M, \omega_\varphi^\alpha), \quad \sigma_\alpha = \sigma(M, \omega_\varphi^\alpha).$$

Now, observe that

$$\mathcal{L}_\alpha^2 = 2\chi_\alpha + 3\sigma_\alpha = \frac{1}{4\pi^2} \int \left(2|W_+|^2 + \frac{s_{\omega_\varphi^\alpha}^2}{24} \right) d\mu,$$

where s , W_+ and W_- are the scalar curvature, the self-dual and anti-self-dual Weyl curvatures of ω_φ^α . Since ω_φ^α is a smooth Kähler metric on $\overline{M} \setminus D$, a *local* computation shows the pointwise equality

$$|W_+|^2 = \frac{s^2}{24}$$

which therefore implies

$$\mathcal{L}_\alpha^2 \leq 3 \left(\frac{1}{4\pi^2} \int \frac{s^2}{24} d\mu \right) \leq \left(\frac{1}{4\pi^2} \int 2|W_-|^2 + \frac{s^2}{24} d\mu \right) = 3(2\chi_\alpha - 3\sigma_\alpha).$$

By letting α approach one, we conclude that

$$\chi(\overline{M}) - \chi(D) \geq 3(\sigma(\overline{M}) - \frac{1}{3}D^2).$$

Moreover, by using the fact that (\overline{M}, D) is a logarithmic surface we have that

$$K_{\overline{M}}^2 = 2\chi(\overline{M}) + 3\sigma(\overline{M}), \quad \chi(D) = -K_{\overline{M}} \cdot D - D^2,$$

which implies

$$(10) \quad 3(\chi(\overline{M}) - \chi(D)) \geq \mathcal{L}^2.$$

The final step is to show the equivalence of 9 and 10. First, let us consider the short exact sequence of sheaves

$$0 \longrightarrow \Omega_{\overline{M}}^1 \longrightarrow \Omega_{\overline{M}}^1(\log D) \longrightarrow \mathcal{O}_D \longrightarrow 0;$$

for more background see again Chapter 3 in [15]. By the Whitney product formula we have

$$c(\Omega_{\overline{M}}^1(\log D)) = c(\Omega_{\overline{M}}^1) \cdot c(\mathcal{O}_D).$$

Since D is reduced and effective, in order to compute $c(\mathcal{O}_D)$ we can simply apply the Whitney product formula to the standard short exact sequence

$$0 \longrightarrow \mathcal{O}_{\overline{M}}(-D) \longrightarrow \mathcal{O}_{\overline{M}} \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

In conclusion, we obtain

$$c(\Omega_{\overline{M}}^1(\log D)) = (1 + c_1(\Omega_{\overline{M}}^1) + c_2(\Omega_{\overline{M}}^1)) \cdot (1 + D + D^2)$$

which implies

$$c_1(\Omega_{\overline{M}}^1(\log D)) = K_{\overline{M}} + D = \mathcal{L}$$

and with a slight abuse of notation

$$c_2(\Omega_{\overline{M}}^1(\log D)) = c_2(\Omega_{\overline{M}}^1) + K_{\overline{M}} \cdot D + D^2 = \chi(\overline{M}) - \chi(D).$$

This concludes the proof of 9 when the boundary divisor D is smooth. \square

It would be extremely interesting to extend this argument in the case when D is reduced with normal crossing. The first problem is that, as shown in Theorem 5.1, not all pairs (\overline{M}, D) for which \mathcal{L} is big, nef and ample modulo D admit a 1-parameter family of negative Kähler-Einstein metrics with small edge singularities along D . Nevertheless, combining Proposition 4.1 with Theorem 6.1. in [11], one can still construct a 1-parameter family of negative Kähler-Einstein metrics γ_φ^α on $\overline{M} \setminus D$. More precisely, given a logarithmic surface (\overline{M}, D) as in Proposition 4.1 and without interior (-2) -curves, for any $\alpha \in [\overline{\alpha}, 1)$ let γ_α be a smooth semi-positive representant for the \mathbb{R} -cohomology class $[\mathcal{L}_\alpha]$. This smooth form is *strictly* positive outside the boundary (-2) -curves intersecting the other components of D in two points only. Now, let Ω be a smooth volume form on \overline{M} and consider the family of degenerate complex Monge-Ampère equations

$$(11) \quad (\gamma_\alpha + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^{f+\varphi} \frac{\Omega}{\prod_i \|\sigma_i\|^{2\alpha}}$$

for any $\alpha \in [\overline{\alpha}, 1)$. By Theorem 6.1. in [11], for a fixed α this degenerate equation admits a unique solution $\varphi \in L^\infty(\overline{M})$ which is smooth on $\overline{M} \setminus D$. Moreover, by appropriately choosing the function f in 11 we can arrange γ_φ^α to be Einstein with negative scalar curvature on $\overline{M} \setminus D$. Thus, the remaining problem is to give a *topological* interpretation of the curvature integrals

$$\chi_\alpha = \chi(M, \gamma_\varphi^\alpha), \quad \sigma_\alpha = \sigma(M, \gamma_\varphi^\alpha).$$

Conjecturally, the same elegant formulas given in Theorem 2.1. and Theorem 2.2. of [1] hold. The proof of 9 should then follow as in the smooth boundary divisor case.

Concluding, a Chern-Weil approach to 9, through deformations of negative Kähler-Einstein metrics with singularities along D , has to rely on a generalization of the recent theory of M.F. Atiyah and C. LeBrun [1] outside the realm of Riemannian metrics with pure cone-edge asymptotic.

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